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MAXIMAL ERGODIC THEOREMS AND APPLICATIONS TO RIEMANNIAN GEOMETRY

BY

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The first author dedicates this paper to his parents José Martiniano and Zoraide

ABSTRACT

We prove new ergodic theorems in the context of infinite ergodic theory, and give some applications to Riemannian and Kähler manifolds without conjugate points. One of the consequences of these ideas is that a complete manifold without conjugate points has nonpositive integral of the infimum of Ricci curvatures, whenever this integral makes sense. We also show that a complete Kähler manifold with nonnegative holomorphic curvature is flat if it has no conjugate points.

0. Introduction

Infinite ergodic theory is the study of measure-preserving transformations of infinite measure spaces. A class of very natural examples is that of null-recurrent Markov chains (resp. their shifts) such as the symmetric coin-tossing random walk on the integers. There is a great variety of ergodic behavior infinite measure-preserving transformations can exhibit, and they have undergone some intense research within the last twenty years, much of which is associated with the name of Aaronson ([Aa]). In his book Aaronson studied the standard σ-finite measure spaces and non-singular measure preserving transformations.

This paper will provide another class of natural examples in the category of infinite ergodic theory. We prove new maximal ergodic theorems, which include

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some known geometric results and have some new geometric consequences. We will restrict the discussion on manifolds even though some of the results can be generalized to more general cases.

Let N be a manifold equipped with the σ-algebra β of Borel sets, a flow T_t, $t \in \mathbb{R}$, and a T_t-invariant measure μ. Let $g: N \to \mathbb{R}$ be a measurable function. We say that a measurable function g has **well-defined integral** if either the positive or the negative part of g is integrable on M. We start with the statement of the classical Maximal Ergodic Theorem (see [Pt], here we use "inf" instead of the usual "sup" for later convenience in applications).

MAXIMAL ERGODIC THEOREM: *Let g be a measurable function, with well-defined integral on N, and $Z \subset N$ be a T_t-invariant Borel subset. Set*

$$E[g] = \left\{ w \in Z \mid \inf_{s>0} \int_0^s g(T_t w)dt \leq 0 \right\}.$$

Then $\int_{E[g]} g d\mu \leq 0$.

Our main ergodic result is the following new maximal ergodic theorem.

THEOREM 0.1: *Let f be a measurable function with well-defined integral on N and $Z \subset N$ be a T_t-invariant Borel set. Consider the following T_t-invariant subset,*

$$E(f) = \left\{ w \in Z \mid \liminf_{\substack{u \to -\infty \\ v \to +\infty}} \int_u^v f(T_t w)dt \leq 0, \ \liminf_{s \to \pm\infty} \int_{I_s} f(T_t w)dt < +\infty \right\},$$

where I_s denotes the interval $[0, s]$ if $0 < s$, and $[s, 0]$ if $s < 0$. Then $\int_{E(f)} f d\mu \leq 0$.

Here we allow the time to go to infinity in both directions and the total measure spaces to be infinite. Thus many results in ergodic theory can be reformulated under this point of view. As shown by our applications to Riemannian geometry, it is particularly useful in spaces of infinite measure.

When the measure of N is finite, it is not difficult to obtain Theorem 0.1 from the Maximal Ergodic Theorem. So the importance of Theorem 0.1 relies on the ergodic theory on infinite measure spaces. We will also give in the first section another version of Theorem 0.1 (see Theorem 1.2) with the conservative and dissipative parts of N separated.

As a corollary of Theorem 0.1 we can obtain a pointwise ergodic Theorem.

COROLLARY 0.1: *Let f be a measurable function with well-defined integral on some T_t-invariant Borel subset E. Assume that for almost all $w \in E$ we have*

$$\liminf_{s \to \pm\infty} \int_{I_s} f(T_t w)dt < +\infty, \ \limsup_{s \to \pm\infty} \int_{I_s} f(T_t w)dt > -\infty.$$

Then

(a) *the limit*

$$\lim_{\substack{u\to-\infty\\ v\to+\infty}} \frac{1}{v-u}\int_u^v f(T_t w)dt := \bar{f}(w)$$

exists almost everywhere in E.

(b) $\bar{f}(T_t w) = \bar{f}(w)$, *almost everywhere in* E;

(c) $\int_E |\bar{f}|d\mu \leq \int_E |f|d\mu$, *hence* $\bar{f}_{|E} \in L^1(E)$ *provided that* $f_{|E} \in L^1(E)$;

(d) *if further* E *has finite measure then* $\int_E f d\mu = \int_E \bar{f} d\mu$, *and*

$$\frac{1}{v-u}\int_u^v f(T_t(.))dt \to \bar{f}(.)$$

in $L^1(E)$ *as* $u \to -\infty$ *and* $v \to +\infty$.

Our second ergodic result is a rigidity theorem corresponding to Theorem 0.1.

THEOREM 0.2: *Let* N *and* f *be as in Theorem 0.1 and let* γ_w *denote the orbit satisfying* $\gamma_w(0) = w$. *Assume that there exists an open set* U *such that the* T_t-*invariant set* $E(f)$ *satisfies* $U \subset E(f) \subset \bar{U}$, *where* $\bar{U}$ *is the closure of* U. *Suppose that* $\int_{E(f)} f d\mu = 0$ *and* f *is continuous. Then* $f \equiv 0$ *on* $E(f)$ *provided that one of the following conditions holds:*

(a) *The interior of* $E(f)$ *has* **few recurrent orbits**, *namely, for each point* $w \in E(f)$ *and each neighbourhood* W *of* w, *there exists a Borel set* $E \subset W$ *with positive measure, such that for each* $w \in E$, *at least one of the two pieces* $\gamma_w|_{t\geq 0}$, $\gamma_w|_{t\leq 0}$ *does not intersect* E *for* $|t|$ *large enough; and for almost all* $w \in E(f)$, *the condition*

$$\liminf_{\substack{u\to-\infty\\ v\to+\infty}} \int_u^v f(T_t w)dt = 0$$

implies that $f(T_t w) \equiv 0$ *along* γ_w.

(b) *There exists a measurable function* x *on* $E(f)$, *such that, for almost all* w, *there exists* $a = a_{\gamma_w} > 0$ *such that the function* $x \circ \gamma$ *satisfies the following inequality of Ricatti type:*

$$(x\circ\gamma)' + a(x\circ\gamma)^2 + f\circ\gamma \leq 0.$$

There are many examples of manifolds with the property in condition (a), for example the unit tangent bundles SM of complete and noncompact manifold M with sectional curvature $K \geq 0$, or a complete simply connected manifold without conjugate points. However, the unit tangent bundle SS^n of the standard sphere and the unit tangent bundle ST^n of a flat torus are typical examples which do not satisfy this property. The conditions in Theorem 0.2 are natural in our geometric applications. We show in Examples 3.1 and 3.2 in the third section that the conditions for rigidity in Theorem 0.2 cannot be dropped.

Our ergodic theorems extend several geometric results. Among them are: the famous theorem of Hopf-Green ([Gn]) which says that the integral of scalar curvature of a closed manifold without conjugate points is nonpositive and is zero if and only if the metric is flat; and its generalization to noncompact Riemannian manifolds by Guimarães ([Gu]). Besides, we also have new geometric applications involving Ricci curvature in Riemannian cases and holomorphic curvature in Kähler cases (see §2).

The rest of this paper is organized as follows. In the first section we prove the ergodic results. In the second section we apply them to Riemannian geometry, particularly to the integral of curvature, and to integral geometry. In the third section we present the examples mentioned in the discussions.

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1. Proof of the ergodic results.

Let $f\colon N \to \mathbb{R}$ be a measurable function, which is assumed to be Lebesgue integrable along compact parts of some orbit starting at $w \in N$. We define:

$$i(w,f) := \liminf_{\substack{u\to-\infty\\ v\to+\infty}} \int_u^v f(T_t w)dt, \quad I(w,f) := \limsup_{\substack{u\to-\infty\\ v\to+\infty}} \int_u^v f(T_t w)dt,$$

$$i_+(w,f) := \liminf_{s\to+\infty} \int_0^s f(T_t w)dt, \quad I_+(w,f) := \limsup_{s\to+\infty} \int_0^s f(T_t w)dt,$$

$$i_-(w,f) := \liminf_{s\to-\infty} \int_s^0 f(T_t w)dt, \quad I_-(w,f) := \limsup_{s\to-\infty} \int_s^0 f(T_t w)dt,$$

and then

$$E(f) = \{w \in Z \mid i(w,f) \le 0,\ i_+(w,f) < +\infty,\ i_-(w,f) < +\infty\}.$$

Let us begin our proof by recalling some facts of ergodic theory. We refer to [Pt] as a basic reference. Given a positive integrable function g on some T_t-invariant

Borel set Z, define the following T_t-invariant sets:

$$D^+ = \left\{ w \in Z | \text{ the Lebesgue integral } \int_0^{+\infty} g(T_t w)dt < +\infty \right\}, \; C^+ = Z \backslash D^+.$$

The Hopf decomposition $Z = D^+ \cup C^+$ does not depend on the integrable positive function g up to a set of measure 0. D^+ and C^+ are called, respectively, the dissipative and the conservative parts of Z. If we consider the reverse flow T_{-t} we obtain another decomposition $Z = D^- \cup C^-$, with similar definitions as above. The set $D = D^+ \cap D^-$ satisfies the Lebesgue integral $\int_{-\infty}^{+\infty} h(T_t w)dt < +\infty$, for every integrable function h, for almost all $w \in D$. Set $T(w) = T_1 w$. Define an equivalence relation on Z by saying $v \sim w$ if there exists $j \in \mathbb{Z}$ so that $T^j(v) = w$. Let Ω be the set of equivalence classes and $\pi: Z \to \Omega$ be the natural projection. We say that a Borel set $E \subset Z$ is a wandering set if $T^j(E) \cap E = \emptyset$, for every $j \geq 1$. Now we consider the σ-algebra on Ω induced by π, in which a set $A \subset \Omega$ is measurable if and only if $\pi^{-1}(A)$ is a Borel set. Let $\tilde{\mu}$ be the measure on Ω given by

$$\tilde{\mu}(\tilde{E}) = \sup\{\mu(E) | \, E \subset \pi^{-1}(\tilde{E}), \; E \text{ is a wandering set}\}.$$

We state here a result of Guimarães which will be used in the proof of Theorem 0.1.

PROPOSITION 1.1 (Proposition 2.3 in [Gu]): *If g is an integrable function on Z, then*

$$\int_D g d\mu = \int_{\pi(D)} \int_{-\infty}^{\infty} g(T_t w) dt d\tilde{\mu}.$$

Now we are ready to prove Theorem 0.1.

Proof of Theorem 0.1: Let f_+ (respectively f_-) be the positive part (respectively, the negative part) of f. By hypothesis either $\int_{E(f)} f_+ d\mu < +\infty$ or $\int_{E(f)} f_- d\mu < +\infty$. So we can assume that $\int_{E(f)} f_- d\mu < +\infty$, otherwise there is nothing to prove. Take any integrable function $0 \leq g \leq f_+$. We will prove that $\int_{E(f)} (g - f_-) d\mu \leq 0$, hence $\int_{E(f)} g d\mu \leq \int_{E(f)} f_- d\mu$. Since g is arbitrary, the monotone convergence theorem implies that f_+ is also integrable and $\int_{E(f)} f_+ d\mu \leq \int_{E(f)} f_- d\mu$, hence $\int_{E(f)} f d\mu \leq 0$. To prove that $\int_{E(f)} (g - f_-) d\mu \leq 0$, we consider the disjoint union $E(f) = D \cup C^+ \cup (C^- \backslash C^+)$.

Take $w \in D$. Since $i(w, f) \leq 0$, there exist sequences $u_k \to -\infty, v_k \to +\infty$ such that $\lim_{k \to +\infty} \int_{u_k}^{v_k} f(T_t w)dt \leq 0$. Since $\int_{-\infty}^{\infty} (g - f_-)(T_t w)dt < +\infty$ we have

$$\int_{-\infty}^{\infty} (g - f_-)(T_t w)dt = \lim \int_{u_k}^{v_k} (g - f_-)(T_t w)dt \leq \lim \int_{u_k}^{v_k} f(T_t w)dt \leq 0,$$

for almost all $w \in D$. Then from Proposition 1.1 we obtain

$$\int_D (g - f_-) d\mu = \int_{\pi(D)} \int_{-\infty}^{\infty} (g - f_-)(T_t w) dt d\tilde{\mu} \leq 0.$$

Now we take $w \in C^+$ and fix some positive integrable function f_0 and $\delta > 0$. Since $i_+(w, f) < +\infty$ there exist $L \in \mathbb{R}$ and a sequence $s_k \to +\infty$ such that $\int_0^{s_k} f(T_t w) dt < L$. So we have

$$\int_0^{s_k} (g - f_- - \delta f_0)(T_t w) dt \leq \int_0^{s_k} f(T_t w) dt - \delta \int_0^{s_k} f_0(T_t w) dt \to -\infty,$$

for almost all $w \in C^+$, by definition of C^+. By the Maximal Ergodic Theorem we have $\int_{C^+} (g - f_- - \delta f_0) d\mu \leq 0$, and we get $\int_{C^+} (g - f_-) d\mu \leq 0$, by making $\delta \to 0$.

In the case that $w \in C_-$ the proof is similar, applying the Maximal Ergodic Theorem to the reverse flow T_{-t}, and using that $i_-(w, f) < +\infty$. This concludes the proof of Theorem 0.1. ∎

Applying Theorem 0.1 to $-f$ we obtain:

THEOREM 1.1: *Let f be a measurable function with well-defined integral on N and $Z \subset N$ be a Borel set, which is T_t-invariant. Consider the following T_t-invariant subset,*

$$E(f) = \{w \in Z \mid I(w, f) \geq 0,\ I_+(w, f) > -\infty,\ I_-(w, f) > -\infty\}.$$

Then $\int_{E(f)} f d\mu \geq 0$.

The equality in Theorem 1.1 implies a rigidity theorem completely similar to Theorem 0.2. We note also that, in the proof of Theorem 0.1, when $w \in D$ we used only $i(w, f) \leq 0$, and when $w \in (C^+ \cup C^-)$ we used only $i_+(w, f) < +\infty$, $i_-(w, f) < +\infty$. Thus we can separate Theorem 0.1 considering the dissipative part D and the conservative parts C^+ and C^-. This modification has some applications (see for example Corollary 2.2) and we state it as the following theorem.

THEOREM 1.2: *Let N, Z and f be as in Theorem 0.1. Consider the Hopf decomposition $Z = D \cup C^+ \cup C^-$. Set: $D(f) = \{w \in D \mid i(w, f) \leq 0\}$, $C^+(f) = \{w \in C^+ \mid i_+(w, f) < +\infty\}$, $C^-(f) = \{w \in C^- \mid i_-(w, f) < +\infty\}$. Then the integral of f is nonpositive on each one of the sets $D(f)$, $C^+(f)$, $C^-(f)$.*

Note that $C^+ \cap C^-$ could be nonempty. So, if it is convenient, we can use $C^- \backslash C^+$ instead of C^- to have a disjoint union (in fact we did it in the proof of Theorem 0.1).

Now we begin to prove our Theorem 0.2 by stating two lemmas about Ricatti inequalities which follow from [Gu], Proposition 3.3. For a positive constant a and a continuous function x, consider the following inequality of Ricatti type:

$$x(t_2) - x(t_1) + a\int_{t_1}^{t_2} x^2(s)ds + \int_{t_1}^{t_2} f(s)ds \leq 0, \quad t_1 < t_2. \tag{1.1}$$

LEMMA 1.1: *If x is a solution of (1.1) for all real numbers $t_1 < t_2$, then we have $\xi := \liminf_{t\to+\infty} \int_{-t}^{t} f(s)ds \leq 0$. Furthermore, we have $\xi = 0$ if and only if $f(t) \equiv 0$ and $x(t) \equiv 0$.*

LEMMA 1.2: *Assume that x satisfies (1.1) for all $t_2 > t_1$. Then, for all $\varepsilon > 0$, we have $\eta = \liminf_{t\to+\infty} \int_{t_1}^{t} f(s)ds < +\infty$.*

Now we are in position to prove Theorem 0.2.

Proof of Theorem 0.2: First we consider the hypothesis (a) in Theorem 0.2. Suppose by contradiction that $\int_{E(f)} f d\mu = 0$ and that there exists $w \in E(f)$ with $f(w) \neq 0$. Since f is continuous, and N has few recurrent orbits, we can assume without loss of generality that w is in the interior of $E(f)$, and that γ_w is not constant. So there exists a neighbourhood W of w of the form $[0,\varepsilon] \times B$, where B is an open disk and W is given by the Theorem of the Tubular Flow. We can assume also that $f \neq 0$ on W, and that ε is the time that each connected component of an orbit remains on W. Let $\tilde{E}$ be the Borel subset of points $x \in W$, such that at least one of the pieces $\gamma_x|_{t>0}$, $\gamma_x|_{t<0}$ does not intersect W for sufficiently large $|t|$. By our hypotheses, we have $i(x, f) < 0$ for almost all $x \in W$, and $\mu(\tilde{E}) > 0$. Let E be the set of points $x \in \tilde{E}$ satisfying $\gamma_x(t) \notin W$ for sufficiently large $t > 0$. We have $\mu(E) > 0$ or $\mu(\tilde{E}\backslash E) > 0$. Without loss of generality we assume that $\mu(E) > 0$ (the other possibility can be treated similarly).

Given $x \in E$, let γ_x^1 be the last connected component of γ_x which enters in W and let $\gamma_x^2, \gamma_x^3, \ldots$ be the preceding components. If x is in γ_x^j we set $j(x) = j$. So we define g: $E(f) \to \mathbb{R}$ given by

$$g(x) = \frac{\max\{i(x,f), -1\}}{2^{j(x)}\varepsilon} \text{ if } x \in E, \quad g(x) = 0 \text{ if } x \in E(f)\backslash E.$$

It is not difficult to see that g is a measurable function. In fact, the function φ: $E \times \mathbb{R}^2 \to \mathbb{R}$ given by $\varphi(x,u,v) = \int_u^v f(T_t x)dt$ is measurable, hence $i(x, f)$ is measurable. By the continuity of the flow T_t, for y in a small neighbourhood of x we have $j(y) \geq j(x)$, so the function j is semi-continuous, hence it is measurable.

Note also that

$$\int_{\gamma_x^j} g(T_t x)dt = \frac{\max\{i(x,f),-1\}}{2^j},$$

hence $0 > \int_{-\infty}^{+\infty} g(T_t x)dt \geq \max\{i(x,f),-1\}$.

So we define $\bar{f} = f - g$. By considering the cases $i(x,f) \leq -1$ and $0 > i(x,f) > -1$ we still have $i(x,\bar{f}) \leq 0$, $i_+(x,\bar{f}) < +\infty$ and $i_-(x,\bar{f}) < +\infty$, for almost all $x \in E(f)$. Then we have $\int_{E(f)} \bar{f} d\mu \leq 0$ by Theorem 0.1. On the other hand, we have

$$\int_{E(f)} \bar{f} d\mu = \int_{E(f)} f d\mu - \int_{E(f)} g d\mu = 0 - \int_{E(f)} g d\mu > 0.$$

This contradiction proves Theorem 0.2 under the hypothesis (a).

Now we assume hypothesis (b) of Theorem 0.2. By integration we arrive at inequality (1.1). Fix $\varepsilon > 0$. First we rewrite the inequality (1.1) in the following form:

$$(1.2)\quad x(t_2) - x(t_1) + a\varepsilon \int_{t_1}^{t_2} x^2(t)dt + \int_{t_1}^{t_2} \big(f(t) + (1-\varepsilon)ax^2(t)\big)dt \leq 0, \quad t_1 < t_2.$$

For almost all $w \in E(f)$ we can apply Lemma 1.1 to (1.2) obtaining

$$i\big(w, f + a(1-\varepsilon)x^2\big) \leq \liminf_{t\to+\infty} \int_{-t}^{t} \{f(T_s w) + a(1-\varepsilon)\big(x(T_s w)\big)^2\} ds \leq 0.$$

Thus we obtain $i(w, f + a(1-\varepsilon)x^2) \leq 0$. By Lemma 1.2 we have

$$i_+\big(w, f + a(1-\varepsilon)x^2\big) < +\infty \quad \text{and} \quad i_-\big(w, f + a(1-\varepsilon)x^2\big) < +\infty.$$

To apply Theorem 0.1 to $f + a(1-\varepsilon)x^2$ we need only check that $f + a(1-\varepsilon)x^2$ has well-defined integral on $E(f)$. Since f has well-defined integral and $\int_{E(f)} f d\mu = 0$, we conclude that $\int_{E(f)} f_+ d\mu = \int_{E(f)} f_- d\mu < +\infty$. Since the negative part of $g := f + a(1-\varepsilon)x^2$ is not bigger than f_-, then $\int g_- < +\infty$ and we can apply Theorem 0.1. We obtain

$$\int_{E(f)} \big(f + a(1-\varepsilon)x^2\big) d\mu = \int_{E(f)} f d\mu + \int_{E(f)} (1-\varepsilon)ax^2 d\mu \leq 0.$$

Since $\int_{E(f)} f d\mu = 0$, we conclude that $x \equiv 0$ almost everywhere. We assert that this implies that $f \leq 0$. If this is not true, the continuity of f implies that there exists an open set W contained in the interior of $E(f)$ such that $f > 0$ on W. By (1.1) we conclude that x is strictly decreasing along the orbit $\gamma_w \cap W$, for almost all $w \in W$. In particular these orbits are not constant, hence W contains a neighbourhood V given by the Theorem of the Tubular Flow. Fix a transversal section Σ in V. For almost any $w \in \Sigma$ the orbit $\gamma_w \cap V$ has $x = 0$ at most at one of its points. This contradicts the fact that $x = 0$ almost everywhere on V. So we have $f \leq 0$. Since $\int_{E(f)} f d\mu = 0$, the continuity of f leads to $f \equiv 0$ on $E(f)$. The proof of Theorem 0.2 is completed. ∎

Proof of Corollary 0.1: The proof is completely similar to the proof of the Birkhoff Pointwise Ergodic Theorem in [Pt], using our Theorem 0.1 instead of the Maximal Ergodic Theorem. We observe that Corollary 2.2 in [Pt], which is needed in the proof, is shown there by assuming finite measure, but it could be easily obtained in the infinite case, just considering a convergence by sets of finite measure. The proof of the Birkhoff Theorem is done there for the case of a measure-preserving transformation T, instead of a flow T_t preserving measure, but the adaptation to our case is trivial and standard. We will only present the proof of item (c), since it is short, and can give some idea of the adaptations needed to prove our corollary. By item (a) we have

$$\bar{f}(v) = \lim_{n \to +\infty} \frac{1}{2n} \int_{-n}^{n} f(T_t v) dt,$$

for almost all $v \in E$. Since

$$\left| \frac{1}{2n} \int_{-n}^{n} f(T_t v) dt \right| \leq \frac{1}{2n} \int_{-n}^{n} |f(T_t v)| dt,$$

it holds that $|\bar{f}| \leq |f|^{-}$ almost everywhere. We can assume that $f \in L^1(E)$, otherwise the inequality in (c) is trivial. Now we use Fatou's Lemma and the fact that μ is invariant under the flow T_t. We have

$$\int_E |\bar{f}| d\mu \leq \int_E |f|^{-} d\mu \leq \liminf_{n \to +\infty} \int_E \left\{ \frac{1}{2n} \int_{-n}^{n} |f(T_t v)| dt \right\} d\mu = \int_E |f| d\mu < +\infty. \quad ∎$$

2. Some applications of the ergodic theorems

Let M^n be a complete n-dimensional Riemannian manifold with the normalized scalar curvature S_M. In 1948, E. Hopf ([Ho]) proved that the integral $\int_M S_M dV \leq 0$ when M^2 is compact and has no conjugate points, and the integral vanishes if and only if the metric is flat. An important generalization of E. Hopf's Theorem is due to Green ([Gn]) who proved that the dimension restriction is superfluous. Combining with the Gauss–Bonnet Theorem, this beautiful result implies that any metric on torus T^2 without conjugate points is flat. The generalization of the result to T^n known as E. Hopf's conjecture was proved by

Burago and Ivanov ([BI]). After an improvement of Innami ([I]), this celebrated theorem of Hopf and Green has been generalized by Guimarães in [Gu] to complete manifolds without conjugate points, under the additional condition that the Ricci curvature has well-defined integral as a function on the unit tangent bundle SM, equipped with the Liouville measure μ.

All the above-mentioned results could be considered as special cases of our ergodic theorems. Besides, we have the following

THEOREM 2.1: *Let $R(p)$ be the infimum of Ricci curvatures at the point p. Assume that M is a complete Riemannian manifold without conjugate points. If $\int_M R(p)dV$ is well defined on M, then*

$$\int_M R(p)dV \leq 0, \tag{2.1}$$

and equality implies that M is flat.

We will state and give some remarks on the main results of this section before we prove them. The advantage of Theorem 2.1 is that the integrability condition is taken on M instead of SM. As shown by Example 3.4 of the third section, the condition that S_M and $R(p)$ have well-defined integral on M does not imply that Ric has well-defined integral on SM. So Theorem 2.1 cannot be obtained from the above geometric results.

We have another consequence in case of complete Kähler manifolds, by considering the holomorphic curvature H as a function on the unit tangent bundle.

THEOREM 2.2: *Let M be a complete Kähler manifold. If M has no conjugate points and its holomorphic curvature has a well-defined integral on SM, then the integral of the scalar curvature is nonpositive, and it vanishes if and only if M is flat.*

It should be remarked that Ricci curvature can be expressed in terms of holomorphic curvature (see [BG] p. 519). From (6.3) in [BG], we can see that holomorphic curvature has a well-defined integral does not imply that Ricci curvature has a well-defined integral. An easy consequence of our Theorem 2.2 is

COROLLARY 2.1: *Let M be a complete Kähler manifold without conjugate points, and nonnegative holomorphic curvature. Then M is flat.*

Remark 2.1: It should be remarked that we don't assume here that M does not contain conjugate points. A result completely similar to Corollary 2.1 can be obtained for Kähler manifolds with H replacing Ric, except that in that situation the equality will imply the stronger conclusion: 'M is flat'.

We begin the proof of our results with the following proposition. First we note that

$$i(w,f) \leq \liminf_{t\to+\infty} \int_{-t}^{t} f(T_s w)ds. \tag{2.2}$$

PROPOSITION 2.1: *Let $\gamma: \mathbb{R} \to M$ be a geodesic without conjugate points. Let us set $w = \gamma'(0)$. We consider any unit vector field $X(t)$ which is parallel along γ and orthogonal to γ'. We define*

$$K(t) = K\big(X(t), \gamma'(t)\big), \ r(t) = \mathrm{Ric}\big(\gamma'(t)\big).$$

For (c) *and* (d) *below we will require only that $\gamma_{|[0,+\infty)}$ does not contain conjugate points to $\gamma(0)$. We have:*

(a) $\liminf_{t\to+\infty} \int_{-t}^{t} K(s)ds \leq 0$ *and the equality implies that* $K(t) \equiv 0$;
(b) $\liminf_{t\to+\infty} \int_{-t}^{t} r(s)ds \leq 0$ *and, if* $\liminf_{t\to+\infty} \int_{-t}^{t} r(s)ds = 0$, *then the Ricci linear operator* $R_{\gamma'}$ *vanishes along* γ *(due to [Gu], extending [CE]);*
(c) $\liminf_{t\to+\infty} \int_{0}^{t} K(s)ds < +\infty$;
(d) $\liminf_{t\to+\infty} \int_{0}^{t} r(s)ds < +\infty$ *(it extends [Am], and can be obtained from the method of [Gu]).*

Remark 2.2: We remark that in [CE] it is proved that $i(w, \mathrm{Ric}) \leq 0$. Item (b) is stated in [Gu] with lim sup instead of lim inf, but the proof leads only to lim inf, as already observed by P. Ehrlich in his review of [Gu] in Mathematical Reviews. Finally, we note that M does not need to be complete in Proposition 2.1.

Proof of Proposition 2.1: We only prove (a), since (b), (c) and (d) can be obtained easily from [Gu]. Let M be a Riemannian manifold of dimension n. Let $\gamma: \mathbb{R} \to M$ be a geodesic without conjugate points. It is well known by [Gn] that for all $t \in \mathbb{R}$ there exists a symmetric linear operator $U(t) : \{\gamma'(t)\}^{\perp} \to \{\gamma'(t)\}^{\perp}$ satisfying the Ricatti equation

$$U' + U^2 + R(.,\gamma')\gamma' = 0. \tag{2.3}$$

We recall that U' is defined by the equation $U'(v(t)) = (Uv(t))'$ where v is any parallel vector field perpendicular to γ'. Fix a unitary vector field $X \perp \gamma'$ parallel along γ. Set $x(t) =< UX, X >$. From (2.3) it is easy to conclude that

$$x' + x^2 + K(X, \gamma') \leq 0.$$

Now item (a) in Proposition 2.1 follows from Lemma 1.1. ∎

To prove Theorem 2.2 and Corollary 2.1 we first prove the following general result. For a measurable function $f\colon SM \to \mathbb{R}$ as in Theorem 0.1 set $S_f(p) = \frac{1}{w_{n-1}} \int_{S\{p\}} f\, dm$, where dm is the Lebesgue measure on the unitary sphere $S\{p\} \subset T_pM$, and w_{n-1} is the $(n-1)$-dimensional volume of $S\{p\}$.

LEMMA 2.1: *Let M^n be a complete manifold with the integral $\int_{SM} f\, d\mu$ well defined on the unit tangent bundle SM. Assume further that for all geodesic γ it holds that $i_+(\gamma', f) < +\infty$, $i_-(\gamma', f) < +\infty$, and $i(\gamma', f) \leq 0$ (resp., $I_+(\gamma', f) > -\infty$, $I_-(\gamma', f) > -\infty$, and $I(\gamma', f) \geq 0$). Then*

$$\int_M S_f dV \leq 0 \left(\text{resp., } \int_M S_f dV \geq 0 \right). \tag{2.4}$$

Assume moreover that equality holds in (2.4), f is continuous, and either:

(a) *M has few recurrent geodesics, and further the condition $i(\gamma'(0), f) = 0$ (resp., $I(\gamma'(0), f) = 0$) implies $f(\gamma'(t)) \equiv 0$ along γ;*

or

(b) *There exists a measurable function x on SM, such that, for almost all w, there is $a = a_{\gamma_w} > 0$, such that $\tilde{x} = x \circ \gamma_w$ satisfies the Riccati inequality:*

$$\tilde{x}' + a\tilde{x}^2 + f \circ \gamma \leq 0$$
$$(\text{ respectively } \tilde{x}' - a\tilde{x}^2 + f \circ \gamma \geq 0).$$

Then $f \equiv 0$.

Proof of Lemma 2.1: Given any filtration of M by bounded Borel sets D_i, we have a corresponding filtration SD_i of SM. By the Fubini Theorem we have

$$\int_{D_i} S_f dV = \frac{1}{w_{n-1}} \int_{SD_i} f\, d\mu. \tag{2.5}$$

So if $\int_{SM} f\, d\mu$ is well defined on SM it follows easily from (2.5) that the integral of S_f on M is also well defined. Lemma 2.1 follows from Theorems 0.1 and 0.2. ∎

Proof of Theorem 2.2: We can assume that there exists a well-defined vector field $X = J\gamma'$ on SM which is parallel along geodesics. We note that the function $x = \langle UX, X\rangle$ is measurable since U is measurable by [Gn]. It is well known by a result of Berger ([B]) that

$$S_M = \frac{n+1}{2(2n-1)} S_H,$$

where n is the complex dimension of M. So Theorem 2.2 is a direct consequence Lemma 2.1 together with Proposition 2.1, item (a). Note that it is well known that $K(w, Jw) \equiv 0$ implies that M is flat. ∎

Proof of Corollary 2.1: It follows directly from Theorem 0.2 and from the well-known fact that $H \equiv 0$ implies that M is flat. ∎

Now we prepare the proof of Theorem 2.1. When we have $f\colon M \to \mathbb{R}$, we can define $\bar{f}\colon SM \to \mathbb{R}$ by $\bar{f}(p, v) = f(p)$. In this case we have $S_{\bar{f}} = f$.

LEMMA 2.2: *Let $f, \bar{f}$ be as above. If f has well-defined integral on M then $\bar{f}$ has well-defined integral on SM.*

Proof: We have that the negative or the positive part of f is integrable. For example, let us assume that the negative part of f is integrable. Consider a filtration D_i of M by compacts sets D_i. Note that $(f_-)^- = (\bar{f})_-$. So we have

$$\int_{D_i} f_- dV = \int_{D_i} S_{\left((f_-)^-\right)} dV = \frac{1}{w_{n-1}} \int_{SD_i} (\bar{f})_- d\mu \to \int_M f_- dV.$$

This implies that $(\bar{f})_-$ is integrable. So we conclude that the integral of $\bar{f}$ is well defined on SM. ∎

Proof of Theorem 2.1: Consider $R\colon M \to \mathbb{R}$ given by $R(p) = \inf_{v \in S\{p\}} \operatorname{Ric}(v)$. If we assume that R has well-defined integral on M we conclude by Lemma 2.2 that $\bar{R}$ has well-defined integral on SM. If M does not contain conjugate points, then for all $w \in SM$ Proposition 2.1 (b), (d) imply that

$$i(w, \bar{R}) \leq i(w, \operatorname{Ric}) \leq 0,$$

and

$$i_+(w, \bar{R}) \leq i_+(w, \operatorname{Ric}) < +\infty, \quad i_-(w, \bar{R}) \leq i_-(w, \operatorname{Ric}) < +\infty.$$

For the right inequality above we just used that $\operatorname{Ric}(w) = \operatorname{Ric}(-w)$. So we can apply Lemma 2.1 obtaining

$$\int_M R dV = \int_M S_{\bar{R}} dV \leq 0.$$

Now assume that equality holds in the above inequality. Since

$$\operatorname{trace}(U^2) \geq \frac{1}{n-1} (\operatorname{trace}(U))^2,$$

for $y = \text{trace}(U)$ we can easily obtain from (2.3) that

$$y(t_2) - y(t_1) + \frac{1}{n-1}\int_{t_1}^{t_2} y^2(t)dt + \int_{t_1}^{t_2} \bar{R}(\gamma'(t))dt$$
$$\leq y(t_2) - y(t_1) + \frac{1}{n-1}\int_{t_1}^{t_2} y^2(t)dt + \int_{t_1}^{t_2} \text{Ric}(\gamma'(t))dt \leq 0, \quad t_1 < t_2.$$

So Lemma 2.1 implies that $\bar{R} \equiv 0$, hence $i(w, \bar{R}) = i(w, \text{Ric}) = 0$, for all $w \in SM$. So Proposition 2.1 (b) implies that M is flat. Theorem 2.1 is proved. ∎

It is our belief that our results could be generalized to a more general version, so that it could include Huber's theorem as a special case and thus give a new proof of it. To this direction we would need to do some work in comparison of Morse index between Riemannian manifolds.

To show the wideness of applications of Theorems 0.1 and 0.2 we finish this section with an application to the integral geometry on $\mathbb{R}^n$.

COROLLARY 2.2: *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function with well-defined integral. Assume that along each line γ we have*

$$\liminf_{\substack{u\to-\infty \\ v\to+\infty}} \int_u^v f\circ\gamma(t)dt \leq 0. \tag{2.6}$$

Then $\int_{\mathbb{R}^n} f dV \leq 0$. Now we assume that $\int_{\mathbb{R}^n} f dV = 0$, and that the equality in (2.6) implies that $f\circ\gamma \equiv 0$. Then $f \equiv 0$.

Proof: Lemma 2.2 implies that f has well-defined integral on $S\mathbb{R}^n$. The inequality follows from Theorem 1.2, since $S\mathbb{R}^n$ is dissipative (note that all geodesics are nonrecurring). The rigidity part follows from item (a) in Theorem 0.2, since there is no recurrent geodesic. ∎

3. Examples

The examples below show that the conditions on Theorem 0.2 are essential.

Example 3.1: Let S^n be the standard sphere. Consider in SS^n a neighbourhood W of a simple closed orbit $\gamma\colon [0, 2\pi] \to SS^n$, and we may assume that W is of the form $\gamma \times B$, where B is a closed disk and $\gamma \times \{x\}$ is a simple closed orbit $\sigma\colon [0, 2\pi] \to SS^n$, where $\sigma(0) = (\gamma(0), x)$. Take a smooth bump function $\alpha\colon B \to [0, 1]$ with $\alpha(x) > 0$ on the interior of B and $\alpha \equiv 0$ on the boundary ∂B.

Consider $f\colon SS^n \to \mathbb{R}$, given by $f(\gamma(t), x) = \alpha(x)\sin(t)$, for $\big(\gamma(t), x\big) \in W$, and $f \equiv 0$ outside W. We have

$$\int_{-\pi/4-2k\pi}^{2k\pi} \sin t dt < 0,$$

for all $k \in \mathbb{N}$, hence $i(f) \leq 0$, and of course we have $i_+(f) < +\infty$ and $i_-(f) < +\infty$. So we conclude that $\int_{SS^n} f d\mu \leq 0$. Similarly we get $\int_{SS^n} f d\mu \geq 0$, hence $\int_{SS^n} f d\mu = 0$ and $f \not\equiv 0$.

Remark 3.1: The above example and Theorem 0.2 show that, given $a > 0$, there exists no absolutely continuous function $x\colon \mathbb{R} \to \mathbb{R}$ satisfying the inequality $x' + ax^2 + \sin t \leq 0$ almost everywhere. We used the function $\sin t$, but we could have used any other periodic function with null integral on the period. So we conclude that such a function could not satisfy the Ricatti inequality (1.1). In particular, we obtain the following corollary. Finally, we would like to observe that the condition that f is continuous in Theorem 0.2 is clearly essential, otherwise we could modify the null function in a set of measure 0 and get a contradiction.

COROLLARY 3.1: *Let γ be a closed geodesic of length T in some manifold M. Assume that γ does not contain conjugate points. Then either $K(\gamma', v) = 0$ for all vectors v orthogonal to γ or $\int_0^T \mathrm{Ric}\big(\gamma'(t)\big)dt < 0$.*

Proof: By Proposition 2.1 (b) it is easy to see that $\int_0^T \mathrm{Ric}\big(\gamma'(t)\big)dt \leq 0$. If the integral is equal to 0 and $\mathrm{Ric}\big(\gamma'(t)\big) \not\equiv 0$, we can apply the arguments in the above remark and get a contradiction. So we conclude that Ricci curvature vanishes along γ. By Proposition 2.1 (b) again we obtain that $K(\gamma', \cdot) \equiv 0$. ∎

The sphere has many closed geodesics, so we could think that the condition "N has few recurrent orbits" could be relaxed to something like "N has few closed orbits". The following example shows that this is not the case.

Example 3.2: Consider any complete manifold N, with a flow T_t, $t \in \mathbb{R}$, and a T_t-invariant measure μ. Take a nonsingular point $w \in N$ and the nontrivial orbit γ_w. Consider some small neighbourhood W of w given by the Theorem of the Tubular Flow, and we may assume that W is of the form $\gamma \times B$, where $\gamma = \gamma_w$ restricted to $[-\varepsilon, \varepsilon]$, B is a closed disk and $\gamma \times \{x\}$ is an orbit $\sigma\colon [-\varepsilon, \varepsilon] \to N$, where $\sigma(0) = \big(\gamma(0), x\big)$. Finally we assume that, for almost every orbit σ passing through W, it returns infinitely many times to W in both directions (for example, if $N = ST^n$, for a flat torus T^n). Take a smooth bump function $\alpha\colon B \to [0, 1]$ with

$\alpha(x) > 0$ on the interior of B and $\alpha \equiv 0$ on the boundary ∂B. Consider $f\colon N \to \mathbb{R}$, given by $f(\gamma(t), x) = \alpha(x)\sin(\pi t/\varepsilon)$, for $(\gamma(t), x) \in W$, and $f \equiv 0$ outside W. Fix an orbit $\sigma\colon \mathbb{R} \to N$ passing infinitely many times in both directions through W. Set $u = \sigma(0) = (\gamma(0), x_0)$. Consider sequences $u_k \to -\infty, v_k \to +\infty$, such that $\sigma(u_k)$ is of the form $(\gamma(-\varepsilon), x_k)$, $x_k \in B$, and $\sigma(v_k)$ is of the form $(\gamma(\varepsilon/2), y_k)$, $y_k \in B$. Thus we have

$$\int_{u_k}^{v_k} f(T_t u)dt = \int_{u_k}^{v_k - 3\varepsilon/2} f(T_t u)dt + \int_{v_k-3\varepsilon/2}^{v_k} f(T_t u)dt = \int_{v_k-3\varepsilon/2}^{v_k} f(T_t u)dt < 0.$$

This implies that $i(u, f) < 0$. It is not difficult to see that $i_+(\sigma(-\varepsilon), f) \le 0$ and $i_-(\sigma(-\varepsilon), f) \le 0$, hence $i_+(u, f) < +\infty$ and $i_-(u, f) < +\infty$. So we conclude that $\int_N f d\mu \le 0$. Similarly we get $\int_N f d\mu \ge 0$, hence $\int_N f d\mu = 0$ and $f \not\equiv 0$.

The following example shows that the hypothesis $U \subset E(f) \subset \bar{U}$ for some open set U cannot be omitted in Theorem 0.2.

Example 3.3: Consider a flat cylinder $C = S^1 \times \mathbb{R}$. Let v be a unit vector tangent to $\{x\} \times \mathbb{R}$ at $(x, 0)$, for some $x \in S^1$. Take a small closed neighbourhood $\tilde{U} \subset SC$ of v and let U be the image of $\tilde{U}$ under the geodesic flow T_t. $\tilde{U}$ is chosen sufficiently small in order to have $S\{(y, 0)\} \cap U = \emptyset$, where y is the antipodal point of $x \in S^1$. For a sufficiently small neighbourhood V of $(y, 0)$ we still have $SV \cap U = \emptyset$. Remove $(y, 0)$ from C and modify the metric of C in V, in order to obtain a complete manifold $N = C\backslash\{(y, 0)\}$. Since N has now one more end, there is a line σ in N joining the new end and one of the other ends. We have $\sigma \cap U = \emptyset$. Now we consider the set $Z = U \cup \sigma \subset SN$. It is easy to see that Z is invariant under the geodesic flow and that $\int_Z \mathrm{Ric}\, d\mu = 0$, since σ has measure 0. However, we don't have that Z is flat. Note that if we consider the restriction of Ric to Z, we have conditions (a) and (b) of Theorem 0.2 being satisfied for $Z = E(\mathrm{Ric})$, so this example shows that the hypothesis $U \subset E(f) \subset \bar{U}$ is necessary in Theorem 0.2.

The last example shows that the scalar curvature S_M and the function in Theorem 2.1, $R(p)$, may have well-defined integral, while Ric does not have well-defined integral on the unit tangent bundle SM.

Example 3.4: Let $M = \mathbb{C}H^4 \times S^2$, where $\mathbb{C}H^4$ is the complex space form with holomorphic curvature -4 and S^2 is the standard Euclidean sphere of constant curvature 1. It is straightforward to obtain that $S_M \equiv -78$. For v tangent to S^2 we have $\mathrm{Ric}(v) = 1$, and for v tangent to $\mathbb{C}H^4$ we have $\mathrm{Ric}(v) = -10$ and the function in Theorem 2.1 $R(p) \equiv -10$. By using tubular neighbourhoods on SM

and the symmetry of M it is easy to see that both Ric_+ and Ric_- have infinite integral.

References

[Aa] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, Mathematical Surveys and Monographs, 50, American Mathematical Society, Providence, RI, 1997.

[Am] W. Ambrose, *A Theorem of Myers*, Duke Mathematical Journal **24** (1957), 345–348.

[B] M. Berger, *Sur quelques variétés riemanniennes compactes d'Einstein* (French), Comptes Rendus de l'Académie des Sciences, Paris **260** (1965), 1554–1557.

[BG] R. L. Bishop and S. I. Goldberg, *Some implications of the generalized Gauss–Bonnet theorem*, Transactions of the American Mathematical Society **112** (1964), 508–535.

[BI] D. Burago and S. Ivanov, *Riemannian tori without conjugate points are flat*, Geometric and Functional Analysis **4** (1994), 259–269.

[CE] J. Chicone and P. Ehrlich, *Line integration of Ricci curvature and conjugate points in Lorentzian and Riemannian manifolds*, Manuscripta Mathematica **31** (1980), 297–316.

[Gn] L. W. Green, *A theorem of E. Hopf*, Michigan Mathematical Journal **5** (1958), 31–34.

[Gu] F. F. Guimarães, *The integral of the scalar curvature of complete manifolds without conjugate points*, Journal of Differential Geometry. **36** (1992), 651–662.

[Ho] E. Hopf, *Closed surfaces without conjugate points*, Proceedings of the National Academy of Sciences of the United States of America **34** (1948), 47–51.

[I] N. Innami, *Manifolds without conjugate points and with integral curvature zero*, Journal of the Mathematical Society of Japan **41** (1989), 251–261.

[Pt] K. Petersen, *Ergodic Theory*, Cambridge University Press, Cambridge, 1983.